Modelli 1 @ Clamfim

Teoria della misura

Lezione 15

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Una curiosità su Ramanujan e la funzione Gamma

$$\sum_{n=0}^{\infty} (-1)^n \frac{\Gamma^3(n+\frac{1}{2})}{2(n!)^3}(4n+1) = \sqrt{\pi}$$

Euler Reflexion Formula

$$\boldsymbol{\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}}$$

for $x \notin \mathbb{Z}$

Euler Beta Theorem If $x,\, y > 0$:

$$\frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)} = \int_0^1 s^{\,x-1}\,(1-s)^{\,y-1}\mathrm{d}s$$

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This identity can be reformulated introducing the Euler Beta function:

$$\mathrm{B}(x,y) = \int_0^1 s^{\,x-1}(1-s)^{\,y-1}\mathrm{d}s$$

in such a way

$$\mathrm{B}(x,y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}$$

Proof

We start from Gamma's definition $\Gamma(x) = \int_0^{+\infty} t^{x-1} e^{-t} \mathrm{d}t$ then change variable putting $t = u^2$ so that

$$\boldsymbol{\Gamma(x) = 2\int_0^{+\infty} u^{2x-1} e^{-u^2} \mathrm{d}u}$$

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Similarly

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Now use Fubini's Theorem

$$\Gamma(x)\Gamma(y) = 4 \iint_{[0,+\infty)\times[0,+\infty)} u^{2x-1} v^{2y-1} e^{-(u^2+v^2)} \mathrm{d}u\mathrm{d}v$$

Change to polar coordinate

$$\begin{cases} u = \rho \cos \vartheta \\ v = \rho \sin \vartheta \end{cases}$$

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$$\begin{aligned} \Gamma(x)\Gamma(y) &= 4\left(\int_0^{+\infty} \rho^{2x+2y-1} e^{-\rho^2}\,\mathrm{d}\rho\right)\left(\int_0^{\pi/2} \cos^{2x-1}\vartheta\,\sin^{2y-1}\vartheta\,\mathrm{d}\vartheta\right) \\ &= \Gamma(x+y)\int_0^{\pi/2} 2\cos^{2x-1}\vartheta\,\sin^{2y-1}\vartheta\,\mathrm{d}\vartheta \end{aligned}$$

To end the proof we have to show that

$$\mathrm{B}(\boldsymbol{x}, \boldsymbol{y}) = \int_0^{\pi/2} 2 \cos^{2x-1} \vartheta \sin^{2y-1} \vartheta \, d\vartheta$$

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But coming back to Beta's definition

$$\mathrm{B}(x,y) = \int_0^1 s^{x-1}\,(1-s)^{y-1}\mathrm{d}s$$

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Conseguenza

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Show, using Gamma and Beta functions, that the Lebesgue measure of $A = \{(x, y) \in \mathbb{R}^2 \mid x^{2/3} + y^{2/3} \leq 1\}$ is $\dfrac{3}{8}\sqrt{\pi}$

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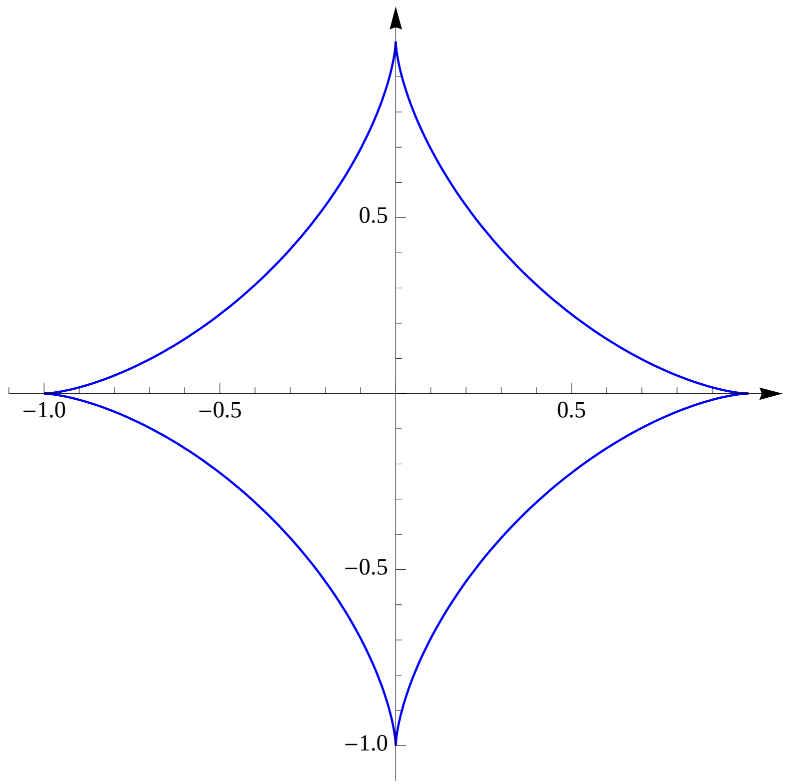

Figure 1: Asteroide

$$m(A) = 4\int_0^1 \left(1 - x^{2/3}\right)^{3/2} \mathrm{d}x$$

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Evaluate using Beta function

$$\int_{\mathbf{0}}^{\infty} \frac{\mathbf{d}\boldsymbol{x}}{\mathbf{1} + \boldsymbol{x}^4}$$

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(Smart) Change of variable

$$1 + x^4 = \frac{1}{u}$$

$$x = \left(\frac{1-u}{u}\right)^{\frac{1}{4}} \implies \mathrm{d}x = -\frac{1}{4}u^{-\frac{5}{4}}(1-u)^{-\frac{3}{4}}\,\mathrm{d}u$$

Thus

$$\int_0^\infty \frac{\mathrm{d}x}{1+x^4} = \frac{1}{4}\int_0^1 u^{-\frac{1}{4}}(1-u)^{-\frac{3}{4}}\mathrm{d}u$$

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Conclusion

$$\int_0^\infty \frac{\mathrm{d}x}{1+x^4} = \frac{1}{4}\mathrm{B}\left(\frac{3}{4},\frac{1}{4}\right) = \frac{1}{4}\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)$$

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Eventually using the reflexion formula $\Gamma(x)\Gamma(1-x) = \dfrac{\pi}{\sin \pi x}$ we get

$$\int_0^\infty \frac{\mathrm{d}x}{1+x^4} = \frac{1}{4}\frac{\pi}{\sin\frac{\pi}{4}} = \frac{\sqrt{2}}{4}\pi$$

Duplication Formula

It is possible to show that

$$\mathbf{2^{2x-1}\Gamma(x)\Gamma(x+\tfrac{1}{2}) = \sqrt{\pi}\,\Gamma(2x)}$$

Tonelli's Theorem Let $A \subset \mathbb{R}^m$ measurable and $f : A \to \overline{\mathbb{R}}$ measurable. Define $\mathcal{S}$ as the $\mathbb{R}^p$ subset where q-sections of A have positive measure i.e.:

$$\mathcal{S} = \{x \in \mathbb{R}^p \mid \ell_q(A_x) > 0\}$$

Then the equality holds

$$\int_A |f(x, y)| \, \mathrm{d}x\mathrm{d}y = \int_S \left(\int_{A_x} |f(x, y)| \, \mathrm{d}y \right) \mathrm{d}x$$

Black Scholes equation:

$$\boxed{\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0, \qquad S \geq 0, \quad t \in [0,T],} \tag{1}$$

where $V(S,t)$ is the value of the option, S the price of the underlying, t the time, T the expiration date, σ the volatility of the underlying and r the risk-free interest rate. The BS Eq. (1) is a linear parabolic equation of the form

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + a\frac{\partial v}{\partial x} + bv, \tag{2}$$

which can always be reduced to a diffusion equation

$$\frac{\partial h}{\partial t} = \frac{\partial^2 h}{\partial x^2}. \tag{3}$$

Then an integral form of the solution of the diffusion Eq. (3) can be obtained for any initial condition using for example Fourier transform methods.

Fourier Transform

From: **The Fourier Transform and its Applications** lecture notes of professor Brad Osgood Stanford University

Let f a real function of a real variable. The Fourier Transform (FT) of f is the complex valued function:

$$\boldsymbol{\mathcal{F} f(s) := \int_{-\infty}^{+\infty} e^{-2\pi\, ist} f(t) \mathrm{d}t}$$

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There is another notation in use

$$\boldsymbol{\hat{f}(s) = \int_{-\infty}^{+\infty} e^{-2\pi\, ist} f(t) \mathrm{d}t}$$

A warning on definitions Our definition of the Fourier transform is a standard one, but it's not the only one. The question is where to put the 2π: in the exponential, as we have done; or perhaps as a factor out front; or perhaps left out completely. There's also a question of which is the Fourier transform and which is the inverse, i.e., which gets the minus sign in the exponential.

Summary provided by T. W. Körner Fourier Analysis:

$$\boldsymbol{\mathcal{F}f(s) = \frac{1}{A}\int_{-\infty}^{+\infty} e^{iBst} f(t)\mathrm{d}t}$$

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The choices that are found in practice are

$$A = \sqrt{2\pi} \qquad B = \pm 1$$

$$A = 1 \qquad B = \pm 2\pi$$

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The definition we've chosen has $A = 1$ and $B = -2\pi$

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The Fourier inversion theorem states that

$$\mathcal{F}(\mathcal{F}^{-1}g) = g, \qquad \mathcal{F}^{-1}(\mathcal{F}f) = f$$

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Plancherel Theorem If $f \in \mathcal{L}^1(\mathbb{R}) \cap \mathcal{L}^2(\mathbb{R})$ then $\hat{f} \in \mathcal{L}^2(\mathbb{R})$ and

$$\int_{-\infty}^{\infty} |f(t)|^2 \mathrm{d}t = \int_{-\infty}^{\infty} |\hat{f}(s)|^2 \mathrm{d}s$$

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Theorem If $f,\ g \in \mathcal{L}^1(\mathbb{R})$ then

$$\int_{-\infty}^{\infty} \hat{f}(s) g(s) \mathrm{d}s = \int_{-\infty}^{\infty} f(x) \hat{g}(x) \mathrm{d}x$$

Example 1: Odd and Even functions.

For any even function $f(t) = f(-t)$ the following relation holds

$$\mathcal{F}f(s) = \int_{-\infty}^{+\infty} \cos(2\pi st) f(t) \mathrm{d}t = 2\int_{0}^{+\infty} \cos(2\pi st) f(t) \mathrm{d}t$$

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If $f(t) = -f(-t)$ is odd, we have

$$\mathcal{F}f(s) = -i\int_{-\infty}^{+\infty} \sin(2\pi st) f(t) \mathrm{d}t = -2i\int_{0}^{+\infty} \sin(2\pi st) f(t) \mathrm{d}t$$

Example 2: Exponential even function Let $f(t) = e^{-|t|}$ then

$$\begin{aligned} \mathcal{F}f(s) &= 2\int_0^{+\infty} \cos(2\pi st) f(t) \mathrm{d}t \\ &= 2\int_0^{+\infty} e^{-t} \cos(2\pi st) \mathrm{d}t \end{aligned}$$

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Now using Euler formulae

$$\boldsymbol{\cos\alpha = \frac{e^{i\alpha}+e^{-i\alpha}}{2} \qquad \sin\alpha = \frac{e^{i\alpha}-e^{-i\alpha}}{2i}}$$

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we find

$$\mathcal{F}f(s) = 2\int_0^{+\infty} e^{-t} \cos(2\pi st) \mathrm{d}t = 2\int_0^{+\infty} e^{-t} \frac{e^{i(2\pi st)} + e^{-i(2\pi st)}}{2}$$

$$\mathcal{F}f(s) = \int_0^{+\infty} \left(e^{(-1+2i\pi s)t} + e^{(-1-2i\pi s)t} \right) \mathrm{d}t$$

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\mathcal{F}f(s) &= \int_0^{+\infty} \left(e^{(-1+2i\pi s)t} + e^{(-1-2i\pi s)t} \right) \mathrm{d}t \\
&= \left[\frac{e^{(-1+2i\pi s)t}}{-1+2i\pi s} \right]_{t=0}^{t=\infty} + \left[\frac{e^{(-1-2i\pi s)t}}{-1-2i\pi s)} \right]_{t=0}^{t=\infty}
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&= \frac{2}{1+4\pi^2 s^2}
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since $\displaystyle\int e^{-t}\cos(At)\mathrm{d}t = \frac{e^{-t}(A\sin At - \cos At)}{1 + A^2}$

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$$\mathcal{F}f(s) = 2\int_0^{+\infty} \cos(2\pi st) f(t)\mathrm{d}t$$
$$= 2\int_0^{+\infty} e^{-t}\cos(2\pi st)\mathrm{d}t$$
$$= \left[\frac{2e^{-t}(2\pi s\sin(2\pi st) - \cos(2\pi st))}{4\pi^2 s^2 + 1}\right]_{t=0}^{t=+\infty}$$

since $\int e^{-t}\cos(At)\mathrm{d}t = \dfrac{e^{-t}(A\sin At - \cos At)}{1+A^2}$ so that

$$\mathcal{F}f(s) = \frac{2}{1+4\pi^2 s^2}$$